

Maths Boot Camp — Session 3

Optimisation and Integrals

Formula & Definition Cheat Sheet — keep this to hand during class

1. Derivatives

Definition — rate of change of f at a point a (slope of the tangent line):

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Notation (all mean the same thing, for $y = f(x)$):

- **Prime (Lagrange):** $f'(x)$
- **Leibniz:** $\frac{dy}{dx}$ or $\frac{df(x)}{dx}$
- **Euler:** $D_x f(x)$ or $Df(x)$
- **Partial** (multivariable — holds every other variable fixed): $\frac{\partial z}{\partial x}$
or $\frac{\partial f(x, y)}{\partial x}$

Differentiation rules:

Rule

Constant: $\frac{d}{dx}[c] = 0$

Constant multiple: $\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$

Sum: $\frac{d}{dx}[f+g] = f' + g'$

Difference: $\frac{d}{dx}[f-g] = f' - g'$

Power: $\frac{d}{dx}[x^n] = nx^{n-1}$

Product: $\frac{d}{dx}[fg] = f'g + fg'$

Quotient: $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - fg'}{g^2}$

Chain: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

2. Optimisation

Finding the x that **maximises** or **minimises** $f(x)$, possibly subject to a constraint.

Second derivative test — at a critical point where $f'(x) = 0$:

- $f''(x) < 0$: bends downward (dome) $\Rightarrow$ **maximum**
- $f''(x) > 0$: bends upward (bowl) $\Rightarrow$ **minimum**

Recipe — **unconstrained** (one variable):

1. Differentiate: find $f'(x)$
2. Set $f'(x) = 0$ and solve for the critical point(s)
3. Check $f''(x)$ at each critical point to classify max/min
4. Substitute back into $f(x)$ for the optimal value

E.g. maximise $f(x) = -2x^2 + 4x + 1$: $f'(x) = -4x + 4 = 0 \Rightarrow x = 1$; $f''(x) = -4 < 0 \Rightarrow$ max; $f(1) = 3$.

Recipe — **constrained** (two variables, one constraint):

1. Use the constraint to write one variable in terms of the other
2. Substitute into the objective (now one variable)
3. Differentiate, set to zero, solve
4. Check f'' , then back-substitute for both variables and the optimal value

E.g. maximise $f(x, y) = xy$ s.t. $x + y = 10$: $y = 10 - x \Rightarrow f(x) = 10x - x^2$; $f'(x) = 10 - 2x = 0 \Rightarrow x = 5$; $f''(x) = -2 < 0 \Rightarrow$ max; $x = y = 5$, $f = 25$.

3. Integrals

An integral is the **area under a curve** — the accumulation of a quantity.

$$\int_a^b f(x) dx$$

$\int$: integral sign; a, b : limits; $f(x)$: integrand; dx : variable.

Basic integration rules (the reverse of differentiation):

Rule
Constant: $\int c dx = cx + C$
Power ($n \neq -1$): $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
Sum/Difference: $\int [f \pm g] dx = \int f dx \pm \int g dx$
Constant multiple: $\int cf(x) dx = c \int f(x) dx$

Remember the $+C$: differentiation kills constants, so an indefinite integral can't recover which constant was there — always add an arbitrary constant of integration.

Properties of integrals:

- **Linearity:** $\int (af+bg) dx = a \int f dx + b \int g dx$
- **Additivity:** $\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$
- **Reversal of limits:** $\int_a^b f dx = - \int_b^a f dx$
- **Constant:** $\int c dx = cx + C$ (a rectangle, width $\times$ height)
- **Zero interval:** $\int_a^a f dx = 0$

Fundamental Theorem of Calculus:

$$\int_a^b f(x) dx = F(b) - F(a), \quad F'(x) = f(x)$$

Find any antiderivative F of f , then evaluate at the two limits and subtract.

Worked example: $\int (3x^2 + 2x + 1) dx = x^3 + x^2 + x + C$.

Definite example: $\int_0^8 (100 - 5t) dt = [100t - 2.5t^2]_0^8 = 800 - 160 = 640$.

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