

Maths Boot Camp — Session 2

Scalars, Vectors, Matrices, and Limits

Formula & Definition Cheat Sheet — keep this to hand during class

1. Scalars & Vectors

Scalar: a single number (magnitude only), e.g. $c = 3$.

Vector: coordinates in space — an ordered list of n numbers places one point in n -dimensional space:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Scalar multiplication (stretches, keeps direction):

$$c\mathbf{v} = \begin{bmatrix} cv_1 & cv_2 & \dots & cv_n \end{bmatrix}$$

Addition / subtraction (entry by entry — same length required):

$$\mathbf{u} \pm \mathbf{v} = \begin{bmatrix} u_1 \pm v_1 \\ \vdots \\ u_n \pm v_n \end{bmatrix}$$

Dot product (multiply corresponding entries, sum — output is a *scalar*, not a vector):

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n = \sum_{i=1}^n u_iv_i$$

Intuition: a scaled, weighted shadow of $\mathbf{v}$ along $\mathbf{u}$.

Euclidean distance — straight-line distance between $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$:

$$d(\mathbf{p}, \mathbf{q}) = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$$

Pythagoras' theorem in disguise; generalises to any number of dimensions by adding more squared-difference terms.

Applied (Downs, 1957): voter $\mathbf{V}_i$, candidate $\mathbf{C}_j$: distance $d_{ij} = \|\mathbf{V}_i - \mathbf{C}_j\|$ (the same formula, written as a vector norm); utility $U_{ij} = -d_{ij}$ — closer candidate $\Rightarrow$ higher utility.

2. Matrices

An array of numbers, m **rows** by n **columns**; entry u_{ij} sits in row i , column j . A vector is just a one-row or one-column matrix.

Addition / subtraction: entry by entry — $\mathbf{A}$ and $\mathbf{B}$ must have the *same dimensions*.

Scalar multiplication: $s\mathbf{A}$ multiplies every entry by s — always possible, regardless of dimensions.

Matrix multiplication: $\mathbf{A}$ ($m \times k$) times $\mathbf{B}$ ($k \times n$) gives $\mathbf{AB}$ ($m \times n$) — the *inner* dimensions must match. Entry (i, j) of $\mathbf{AB}$ is the dot product of row i of $\mathbf{A}$ with column j of $\mathbf{B}$. *Matrix division does not exist.*

Properties:

Property	Holds?
Associative, $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$	Yes
Associative, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$	Yes
Associative, subtraction	No
Commutative, $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$	Yes
Commutative, subtraction	No
Commutative, $\mathbf{AB} = \mathbf{BA}$	No , in general
Distributive (left), $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$	Yes
Distributive (right), $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$	Yes

Because multiplication doesn't commute: keep the same side consistent throughout a calculation (e.g. always multiply $\mathbf{C}$ on the same side).

Special matrix types:

- **Transpose** $\mathbf{A}^T$: rows and columns swapped; $m \times n \rightarrow n \times m$
- **Diagonal:** zeros everywhere except the main diagonal
- **Identity I:** diagonal matrix of all 1's — $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$
- **Symmetric:** equal to its own transpose (mirrors across the diagonal)
- **Zero matrix:** every entry is 0 — the additive identity
- **Triangular:** all entries on one side of the diagonal are 0 (upper or lower)

3. Limits

Neighbourhood: an open interval around a point, e.g. (2, 4) around $p = 3$.

Continuity (informal): drawing a function without raising your pencil.

If $f(x)$ approaches a number L as x moves toward c :

$$\lim_{x \rightarrow c} f(x) = L$$

Working it out: if direct substitution gives $\frac{0}{0}$ (indeterminate), **factor and cancel** first.

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 2$$

Limits at infinity: infinity isn't a number to substitute — ask what value the expression approaches as x keeps growing.

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0 \qquad \lim_{x \rightarrow \infty} \frac{100x}{x + 1} = 100$$

Types of limits:

- **Right-hand:** approaching from larger values
- **Left-hand:** approaching from smaller values
- **Infinity:** as x grows infinitely large
- **Negative infinity:** as x grows infinitely negative

A two-sided limit exists only if the left-hand and right-hand limits agree.

Limit properties (when the individual limits exist):

Rule	Name
$\lim[f + g] = \lim f + \lim g$	Sum
$\lim[\alpha f] = \alpha \lim f$	Scalar multiple
$\lim[fg] = [\lim f][\lim g]$	Product
$\lim\left[\frac{f}{g}\right] = \frac{\lim f}{\lim g}, \lim g \neq 0$	Quotient

Fast workflow: (1) try direct substitution; (2) if $\frac{0}{0}$, factor and cancel; (3) check left-/right-hand limits when needed. $f(c)$ may differ from $\lim_{x \rightarrow c} f(x)$, or may not exist even when the limit does.